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EXPANSION IN EIGENFUNCTIONS OF RELATIONS GENERATED BY PAIR OF OPERATOR DIFFERENTIAL EXPRESSIONS

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Dedicated to blessed memory of Professor Alexander Povzner

Abstract. For relations generated by a pair of operator symmetric differential expressions, a class of generalized resolvents is found. These resolvents are integro-differential operators. The expansion in eigenfunctions of these relations is obtained.

1. The operator differential equation

$$l[y] - \lambda m[y] = m\,[f] \quad t \in \bar{\mathcal{I}}, \quad \mathcal{I} = (a,\ b) \subseteq R^1 \tag{1}$$

is considered on finite or infinite intervals in the space of vector-functions with values in a separable Hilbert space $\mathcal{H}$, where $l[y]$ and $m[y]$ are symmetric operator differential expressions of order r and s respectively, where $r + s > 0$, s is even. Expression $m[y]$ is non-negative and such that an operator first-order system obtained from the homogeneous equation (1) by using quasi-derivatives contains a spectral parameter λ in Nevanlinna's manner.

In the paper, the equation (1) reduces in a special way to a symmetric first order system containing the spectral parameter either in a linear way ($r > s$) or in a nonlinear way ($r \leq s$). Using this reduction and the characteristic operator of this system (see [19], [20]) we construct a class of the generalized resolvents of the minimal relation corresponding to (1). These resolvents are integro-differential operators. From this the inversion formulas and Parseval's equality are obtained. For their proof we modify Strauss's method [27] concerning the case of the generalized resovents as $s = 0$ and $m[y] \equiv y$ which are integral operators (but not integro-differential operators) depending on λ in a more simple way (see [1], [4], [5], [6], [16], [17], [18], [27]) comparing with the case $s > 0$.

The expansion formulas in the solutions of the homogeneous equation (1) were obtained in various particular situations in a number of papers. For $\dim \mathcal{H} = 1$ in the regular case, $r > s$, and for special l and m this was done in [7, 14]. For $\dim \mathcal{H} < \infty$, $m[y] \equiv w(t)y,\ 0 \leq w(t) \in B\,(\mathcal{H})$, the expansion formulas were obtained in [1] for $r = 1$ and, for the general case, in [16]–[18] (see also [26] for the case $r = 1$). Then for $\dim \mathcal{H} < \infty$ the existence of the expansion formulas was proved in [11] under the assumption that the leading coefficient of the expression $m[y]$ is nondegenerate, and the minimal differential operator that corresponds to this expression is uniformly positive on any finite interval (i.e. under these assumptions even the case $m[y] \equiv w(t)y$ with degenerate weight $w(t)$ is not covered). However in [11], as it was mentioned by the authors, the Titchmarsh-Kodaira's formula for an explicit calculation of the spectral matrix is not obtained. Also [11] does not contain an explicit expression for the resolvent and the case $r = s$ is not considered.

Further in the paper, the boundary value problems for the equation (1) with boundary conditions depending on the spectral parameter are considered. We show that for some

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boundary conditions, solutions of these problems are generated by a generalized resolvent if, in contrast to the case $s = 0$, the boundary conditions contain the derivatives of vector-function $f(t)$ that are taken on the ends of interval.

In the general case where $\dim \mathcal{H} < \infty$ we find the absolutely continuous part of the spectral matrix on the axis when the coefficients of the equation (1) are periodic on the semi-axes and also we find the spectral matrix on the semi-axis when the coefficients are periodic. These formulas are obtained using the results that are obtained in [18], [19] for $s = 0$.

Notice that it is not supposed in the paper that for $r \geq s$ the leading coefficient of the expression $m[y]$ (which set the metric) has an inverse in $B(\mathcal{H})$.

Many questions that concern differential operators and relations in the space of vector-functions are considered in the monographs [1, 2, 3, 13, 23, 24, 25, 26] containing an extensive literature. The method of studying these operators and relations based on a use of the abstract Weyl function was proposed in [9].

We denote by $(\ .\ )$ and $\|\cdot\|$ the scalar product and the norm in various spaces with special indexes if it is necessary.

Let an interval $\Delta \subseteq R$, $f(t)$ $(t \in \Delta)$ be a function with values in some Banach space B. The notation $f(t) \in C^l(\Delta)$, $l = 0, 1, \ldots$ (we omit the index l if $l = 0$) means that, in any point of Δ, $f(t)$ has a norm $\|\cdot\|_B$ continuous derivatives of order up to and including l that are taken in the norm $\|\ \|_B$; if Δ is either semi-open or closed interval then on its ends belonging to Δ the one-side continuous derivatives exist. The notation $f(t) \in C_0^l(\Delta)$ means that $f(t) \in C^l(\Delta)$ and $f(t) = 0$ in the neighbourhoods of the ends of Δ.

2. We consider an operator differential equation in a separable Hilbert space $\mathcal{H}_1$,

$$\frac{i}{2}\left((Q(t)x(t))' + Q^*(t)x'(t)\right) - H_\lambda(t)x(t) = W_\lambda(t)F(t), \quad t \in \bar{\mathcal{I}}, \tag{2}$$

where $Q(t)$, $[\Re Q(t)]^{-1}$, $H_\lambda(t) \in B(\mathcal{H}_1)$, $Q(t) \in C^1(\bar{\mathcal{I}})$; the operator function $H_\lambda(t) = H^*_{\bar{\lambda}}(t)$ is continuous in t and is Nevanlinna's in λ. Namely, the following condition holds:

The set $\mathcal{A} \supseteq C\backslash R^1$ exists, any of its points has a neighbourhood independent of $t \in \bar{\mathcal{I}}$, in this neighbourhood $H_\lambda(t)$ is analytic $\forall t \in \bar{\mathcal{I}}$; $\forall \lambda \in \mathcal{A}$ $H_\lambda(t) \in C(\bar{\mathcal{I}})$; the weight $W_\lambda(t) = \Im H_\lambda(t)/\Im\lambda \geq 0$ $(\Im\lambda \neq 0)$.

In view of [20] $\forall \mu \in \mathcal{A} \bigcap R$: $W_\mu(t) = \partial H_\mu(t)/\partial\mu$ is Bochner locally integrable in the uniform operator topology.

For convenience of statements we suppose that $0 \in \bar{\mathcal{I}}$ and we denote $\Re Q(0) = G$.

Let $X_\lambda(t)$ be the operator solution of the homogeneous equation (2) satisfying the initial condition $X_\lambda(0) = I$, where I is an identity operator in $\mathcal{H}_1$.

For any α, $\beta \in \bar{\mathcal{I}}$, $\alpha \leq \beta$ we denote

$$\Delta_\lambda(\alpha, \beta) = \int_\alpha^\beta X^*_\lambda(t)W_\lambda(t)X_\lambda(t)dt,$$
$$N = \{h \in \mathcal{H}_1 \,|\, h \in \operatorname{Ker}\Delta_\lambda(\alpha,\ \beta)\,\forall\alpha,\ \beta\},$$

P is an orthogonal projection onto $N^\perp$. N is independent of $\lambda \in \mathcal{A}$ [20].

For $x(t) \in \mathcal{H}_1$ or $x(t) \in B(\mathcal{H}_1)$ we denote

$$U[x(t)] = ([\Re Q(t)]\,x(t),\, x(t)) \quad \text{or} \quad U[x(t)] = x^*(t)\,[\Re Q(t)]\,x(t),$$

respectively.

As in [20] we introduce the following.

Definition 1. An analytic operator-function $M(\lambda) = M^*(\bar{\lambda}) \in B(\mathcal{H}_1)$ of non-real λ is called a characteristic operator (c.o.) of the equation (2) on $\mathcal{I}$ (or, simply, c.o.), if for $\Im\lambda \neq 0$ and for any $\mathcal{H}_1$-valued vector-function $F(t) \in L^2_{W_\lambda}(\mathcal{I})$ with compact support the corresponding solution $x_\lambda(t)$ of the equation (3) of the form

$$x_\lambda(t, F) = \mathcal{R}_\lambda F = \int_{\mathcal{I}} X_\lambda(t) \left\{ M(\lambda) - \frac{1}{2}\operatorname{sgn}(s-t)(iG)^{-1} \right\} X_{\bar\lambda}^*(s)\, W_\lambda(s)\, F(s)\, ds \tag{3}$$

satisfies the condition

$$(\Im\lambda) \lim_{(\alpha,\beta)\uparrow\mathcal{I}} \left(U[x_\lambda(\beta, F)] - U[x_\lambda(\alpha, F)]\right) \le 0 \quad (\Im\lambda \ne 0). \tag{4}$$

The properties of c.o. and sufficient condition (that are close to necessary condition) of the c.o.'s existence are obtained in [19, 20].

We consider in the separable Hilbert space $\mathcal{H}$ the equation (1), where $l[y]$ and $m[y]$ are symmetric differential expressions of orders r and s correspondingly (one of these orders can be equal to zero), where s is even, with sufficiently smooth coefficients from $B(\mathcal{H})$.

Namely, $l[y] = \sum_{k=0}^{r} i^k l_k[y]$, where $l_{2j} = D^j p_j(t) D^j$, $l_{2j-1} = \frac{1}{2} D^{j-1}\{D q_j(t) + q_j^*(t) D\} D^{j-1}$, $p_j(t) = p_j^*(t)$, $q_j(t) \in B(\mathcal{H})$, $p_j(t), q_j(t) \in C^j(\bar{\mathcal{I}})$, $D = d/dt$; $m[y]$ is defined in a similar way with s instead of r and $\tilde{p}_j(t) = \tilde{p}_j^*(t)$, $\tilde{q}_j(t) \in B(\mathcal{H})$ instead of $p_j(t)$, $q_j(t)$.

We denote by $p(t,\lambda)$ the coefficient at the highest-order derivative in the homogeneous equation (1), i.e.

$$p(t,\lambda) = \begin{cases} p_n(t), & r = 2n > s, \\ p_n(t) - \lambda\widetilde{p}_n(t), & r = s = 2n, \\ -\lambda\widetilde{p}_n(t), & s = 2n > r, \\ i\Re q_{n+1}(t), & r = 2n+1 > s. \end{cases}$$

It is supposed in the paper that for non-real λ, $p^{-1}(t,\lambda) \in B(\mathcal{H})$ for any $t \in \bar{\mathcal{I}}$. We note that in that case where $r = s$ the leading coefficients of both expressions $l[y]$ and $m[y]$ may not have inverses in $B(\mathcal{H})$ (in particular simultaneously) for any $t \in \bar{\mathcal{I}}$.

Denote $p = \max\{r, s\}$ and by $y^{[k]}(t|L)$ we denote the quasi-derivatives [21] of the vector-function $y(t)$ that corresponds to the differential expression L.

Using the substitution

$$x(t) = x(t,\lambda) = \begin{cases} \left(\sum_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \sum_{j=1}^{n} \oplus y^{[p-j]}(t, |l - \lambda m), \\ \qquad \text{if } p = 2n \text{ is even}, \\ \left(\sum_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \left(\sum_{j=1}^{n} \oplus y^{[p-j]}(t, |l - \lambda m)\right) \oplus \left(-iy^{(n)}(t)\right), \\ \qquad \text{if } p = 2n+1 > 1 \text{ is odd}, \\ y(t), \qquad \text{if } p = 1, \end{cases} \tag{5}$$

for t and λ such that $p^{-1}(t,\lambda) \in B(\mathcal{H})$ the equation

$$l[y] - \lambda m[y] = 0, \quad t \in \bar{\mathcal{I}} \tag{6}$$

is reduced to a homogenous equation of type (2) in $\mathcal{H}_1 = \mathcal{H}^p$. Under this substitution for odd $p = r > s$ we formally consider that $s = r - 1$ and if it is necessary we set some leading coefficients in the expression $m[y]$ to be equal to zero. Analogously for even p we formally consider that $r = s$.

Then the quasi-derivatives in (5) are equal to

$$y^{[j]}(t|l - \lambda m) = y^{(j)}(t), \quad j = 0, \ldots, \quad [p/2] - 1, \tag{7}$$

$$y^{[n]}\left(t\,|l-\lambda m\,\right)=\begin{cases}p\left(t,\,\lambda\right)y^{(n)}-\frac{i}{2}\left(q_n-\lambda\tilde{q}_n\right)y^{(n-1)}, & p=2n,\\ -\frac{i}{2}\left(q_{n+1}-\lambda\tilde{q}_{n+1}\right)y^{(n)}, & p=2n+1,\end{cases}\tag{8}$$

$$\begin{gathered}y^{[p-j]}\left(t\,|l-\lambda m\,\right)=-Dy^{[p-j-1]}\left(t\,|l-\lambda m\,\right)+\left(p_j-\lambda\tilde{p}_j\right)y^{(j)}\\ +\frac{i}{2}\left[\left(q^*_{j+1}-\lambda\tilde{q}^*_{j+1}\right)y^{(j+1)}-\left(q_j-\lambda\tilde{q}_j\right)y^{(j-1)}\right],\\ j=0,\ldots,\left[\frac{p-1}{2}\right],\quad q_0\equiv\tilde{q}_0\equiv 0.\end{gathered}\tag{9}$$

With this, $l[y]-\lambda m[y]=y^{[p]}\left(t\,|l-\lambda m\,\right)$.

In the homogeneous equation (2) obtained from equation (6) using substitution (5) for even $p=2n$,

$$\begin{gathered}Q(t)=iJ=\begin{pmatrix}0 & iI_n\\ -iI_n & 0\end{pmatrix},\\ H_\lambda(t)=\left\|h_{\alpha\,\beta}\left(t,\,\lambda\right)\right\|^2_{\alpha,\,\beta=1},\quad h_{\alpha\,\beta}\in B\left(\mathcal{H}^n\right),\end{gathered}\tag{10}$$

where I_n is an identity operator in $B\left(\mathcal{H}^n\right)$, $h_{11}\left(t,\,\lambda\right)=h^*_{11}\left(t,\,\bar{\lambda}\right)$ is a three-diagonal operator matrix whose elements under the main diagonal are equal to

$$\Big(\frac{i}{2}\left(q_1-\lambda\tilde{q}_1\right),\ \ldots\ ,\ \frac{i}{2}(q_{n-1}\lambda\tilde{q}_{n-1})\Big),$$

the elements on the main diagonal are equal to

$$\begin{gathered}\Big(-\left(p_0-\lambda\tilde{p}_0\right),\ldots,\\ -\left(p_{n-2}-\lambda\tilde{p}_{n-2}\right)\frac{1}{4}\left(q^*_n-\lambda\tilde{q}^*_n\right)p^{-1}\left(t,\,\lambda\right)\left(q_n-\lambda\tilde{q}_n\right)-\left(p_{n-1}-\lambda\tilde{p}_{n-1}\right)\Big),\end{gathered}$$

the rest of the elements are equal to zero. $h_{12}\left(t,\,\lambda\right)=h^*_{21}\left(t,\,\bar{\lambda}\right)$ is the operator matrix with identical operators I_1 under the diagonal, the elements on the diagonal are equal to $\left(0,\ \ldots\ ,0,\ -\frac{i}{2}\left(q^*_n-\lambda q^*_n\right)p^{-1}\left(t,\ \lambda\right)\right)$, the rest of the elements are equal to zero. $h_{22}\left(t,\,\lambda\right)=\operatorname{diag}\left(0,\ \ldots\ ,0,\ p^{-1}\left(t,\,\lambda\right)\right)$.

And for odd $p=2n+1$,

$$\begin{gathered}Q(t)=\begin{cases}\begin{pmatrix}0 & iI_n & 0\\ -iI_n & 0 & 0\\ 0 & 0 & q_{n+1}\end{pmatrix}, & p>1,\\ q_1, & p=1,\end{cases}\\ H_\lambda(t)=\begin{cases}\left\|h_{\alpha\,\beta}\left(t,\,\lambda\right)\right\|^2_{\alpha,\,\beta=1}, & p>1,\\ p_0-\lambda\tilde{p}_0, & p=1,\end{cases}\end{gathered}\tag{11}$$

where $B\left(\mathcal{H}^n\right)\ni h_{11}\left(t,\,\lambda\right)=h^*_{11}\left(t,\,\bar{\lambda}\right)$ is a three-diagonal operator matrix whose elements under the diagonal are equal $\left(\frac{i}{2}\left(q_1-\lambda\tilde{q}_1\right),\ldots,\frac{i}{2}\left(q_{n-1}-\lambda\tilde{q}_{n-1}\right)\right)$, the elements on the diagonal are equal to $\left(-\left(p_0-\lambda\tilde{p}_0\right),\ \ldots\ ,-\left(p_{n-1}-\lambda\tilde{p}_{n-1}\right)\right)$, the rest of the elements are equal to zero. $B\left(\mathcal{H}^{n+1},\,\mathcal{H}^n\right)\ni h_{12}\left(t,\,\lambda\right)=h^*_{21}\left(t,\,\bar{\lambda}\right)$ is an operator matrix whose elements with indices $j,\ j-1$ are equal to $I_1,\ j=2,\ \ldots\ ,n$, the element with index $n,\ n+1$ is equal $\frac{1}{2}\left(q^*_n-\lambda q^*_n\right)$, the rest of the elements are equal to zero. $B\left(\mathcal{H}^{n+1}\right)\ni h_{22}\left(t,\,\lambda\right)=h^*_{22}\left(t,\,\bar{\lambda}\right)$ is an operator matrix whose last row is equal to $\left(0,\ \ldots\ ,0,-iI_1,-\left(p_n-\lambda\tilde{p}_n\right)\right)$, the rest of elements are equal to zero.

Therefore in the equation (2) with coefficients (10), (11), $H_\lambda(t)$ depend on λ in a nonlinear manner for $r\le s$, and in a linear manner for $r>s$,

$$H_\lambda(t)=H_0(t)+\lambda H(t),\quad H^*_0(t)=H_0(t).\tag{12}$$

Similarly to the general equation (2), for the equation (2) with coefficients (10),(11), the weight is

$$W_\lambda(t)=\begin{cases}\Im H_\lambda(t)/\Im\lambda, & \text{if } \Im\lambda\neq 0,\\ \dfrac{\partial H_\lambda(t)}{\partial\lambda}, & \text{if } \Im\lambda=0,\quad p^{-1}(t,\lambda)\in B(\mathcal{H}).\end{cases}\tag{13}$$

Everywhere below, unless stated otherwise, we assume that in the equation (2) with coefficients (10), (11), $W_\lambda(t)\geq 0$ ($\Im\lambda\neq 0$).

Moreover, tacitly we assume that the following condition holds:

$$\begin{gathered}\exists\lambda_0\in C;\quad \alpha,\,\beta\in\bar{\mathcal{I}},\quad 0\in[\alpha,\beta],\quad \text{the number } \delta>0:\\ p^{-1}(t,\,\lambda_0)\in B(\mathcal{H}),\quad \forall t\in[\alpha,\beta],\\ m\left[\chi_{\alpha,\beta}y(t,\,\lambda_0),\,\chi_{\alpha,\beta}y(t,\,\lambda_0)\right]\geq\delta\left\|x(0,\,\lambda_0)\right\|^2.\end{gathered}\tag{14}$$

For any solution $y(t,\,\lambda_0)$ of the equation (6) as $\lambda=\lambda_0$, where

$$\begin{gathered}m\left[f(t),\,g(t)\right]=\int_{\mathcal{I}}\sum_{k=0}^{s}m_k\left[f(t),\,g(t)\right]dt,\\ m_{2j}\left[f(t),\,g(t)\right]=\left(\tilde{p}_{2j}(t)f^{(j)}(t),\,g^{(j)}(t)\right),\\ m_{2j-1}\left[f(t),\,g(t)\right]=\tfrac{i}{2}\left\{\left(\tilde{q}_j^*(t)f^{(j)}(t),\,g^{(j-1)}(t)\right)-\left(\tilde{q}_j(t)f^{(j-1)}(t),\,g^{(j)}(t)\right)\right\},\end{gathered}\tag{15}$$

$\chi_{\alpha,\beta}$ is a characteristic function of the interval $(\alpha,\,\beta)$, $x(t,\,\lambda)$ is defined by (5).

For sufficiently smooth vector-function $f(t)$ we denote

$$\mathcal{H}^p\ni F_\lambda(t)=\begin{cases}\left(\sum_{j=0}^{s/2}\oplus f^{(j)}(t)\right)\oplus\mathrm{O}\oplus\ldots\\ \qquad\ldots\oplus\mathrm{O},\ \ r=2n,\ \ r=2n+1>1,\ \ s<2n,\\ \left(\sum_{j=0}^{n-1}\oplus f^{(j)}(t)\right)\oplus\mathrm{O}\oplus\ldots\oplus\\ \qquad\oplus\mathrm{O}\oplus\left(-if^{(n)}(t)\right),\ \ r=2n+1>1,\ \ s=2n,\\ f(t),\ \ r=1,\\ \text{an analog of (5) for } f(t),\ \ r\leq s.\end{cases}\tag{16}$$

Lemma 1. *Let the vector-function* $f(t)\in C^s\left(\bar{\mathcal{I}}\right)$, $F_{\bar\lambda}(t)$ *be defined by* (16) *with* $\bar\lambda$ *instead of* λ, $W_\lambda(t)$ *be defined by* (13), (10), (11). *Then*

$$W_\lambda(t)F_{\bar\lambda}(t)=\begin{cases}\left(\sum_{j=0}^{s/2-1}\oplus\left(f^{[s-j]}(t|m)+\left(f^{[s-j-1]}(t|m)\right)'\right)\right)\oplus\\ \qquad f^{[s/2]}(t|m)\oplus\mathrm{O}\oplus\ldots\oplus\mathrm{O},\\ \qquad r=2n+1,\ \ r=2n,\ \ 0<s<2n,\\ \left(\sum_{j=0}^{n-1}\oplus\left(f^{[s-j]}(t|m)+\left(f^{[s-j-1]}(t|m)\right)'\right)\right)\\ \qquad\oplus\mathrm{O}\oplus\ldots\oplus\mathrm{O}\oplus\left(-if^{[n]}(t|m)\right),\\ \qquad r=2n+1>1,\ \ s=2n,\\ \tilde{p}_0(t)f(t)\oplus\mathrm{O}\oplus\ldots\oplus\mathrm{O},\ \ s=0,\\ \left(\sum_{j=0}^{n-1}\oplus\left(f^{[s-j]}(t|m)+\left(f^{[s-j-1]}(t|m)\right)'\right)\right)\oplus\mathrm{O}\oplus\ldots\\ \qquad\ldots\oplus\mathrm{O}+H_\lambda(t)\left(\mathrm{O}\oplus\ldots\oplus\mathrm{O}\oplus f^{[n]}(t|m)\right),\ \ r\leq s=2n,\end{cases}\tag{17}$$

for λ,t *such that* $p^{-1}(t,\,\lambda)\in B(\mathcal{H})$.

Proof. The proof for $r>s$ follows from (7)–(11), (16).

Let $r\leq s=2n$. Let $\Im\lambda\neq 0$. Since

$$W_\lambda(t)F_{\bar\lambda}=\frac{1}{2i\Im\lambda}\left(\left(H_\lambda(t)F_\lambda(t)-H_{\bar\lambda}(t)F_{\bar\lambda}\right)+H_\lambda(t)\left(F_{\bar\lambda}(t)-F_\lambda(t)\right)\right),\tag{18}$$

using (7)–(10) and the fact that

$$H_\lambda(t)F_\lambda(t)=iJ\left(F_\lambda(t)\right)'-\operatorname{col}\left\{f^{[s]}(t|l-\lambda m),\,0,\,\ldots,\,0\right\},\tag{19}$$

we obtain (17) since $\Im\lambda\neq 0$.

For $\lambda_0 \in R$, $t \in \bar{\mathcal{I}}$ which imply that $p^{-1}(t, \lambda_0) \in B(\mathcal{H})$, formula (17) is proved by passing to the limit for $\lambda \to \lambda_0 + i0$. The lemma is proved. □

As is seen from the proof, Lemma 1 remains true without assuming that $W_\lambda(t) \geq 0\,(\Im\lambda \neq 0)$ and (14).

Denote

$$q = \begin{cases} s/2, & r > s, \\ s, & r \leq s. \end{cases}$$

Lemma 2. *Let the vector-functions* $f(t)$, $g(t) \in C^q\left(\bar{\mathcal{I}}\right)$, $W_\lambda(t)$ *be defined by* (13), (10), (11). *Then*

$$\sum_{k=0}^{s} m_k\,[f(t),\ g(t)] = \left(W_\lambda(t)F_\lambda(t),\ G_\lambda(t)\right)_{\mathcal{H}^P} \tag{20}$$

for $\lambda,\ t$ *such that* $p^{-1}(t, \lambda) \in B(\mathcal{H})$ *(*$F_\lambda(t)$ *is defined by* (16), $G_\lambda(t)$ *is defined in a similar way using* $g(t)$ *) and, therefore,*

$$m\left[\chi_{\alpha,\beta} f(t),\ \chi_{\alpha,\beta}\, g(t)\right] = \left(F_\lambda(t),\ G_\lambda(t)\right)_{L^2_{W_\lambda}(\alpha,\beta)} \tag{21}$$

for $\lambda,\ t$ *such that* $p^{-1}(t,\lambda) \in B(\mathcal{H})$ $\forall t \in [\alpha,\beta] \subseteq \bar{\mathcal{I}}$.

Proof. For $r > s$, (20) follows from (7)–(11), (16). For $r \leq s$, (20) can be proved using (5), (10), (16), (17) and (19). Lemma is proved. □

Note that the proof shows that the formula (20) is valid without the fulfilment of the conditions $W_\lambda(t) \geq 0\,(\Im\lambda \neq 0)$ and (14).

In view of Lemma 2, the left-hand side of (20) is nonnegative for $g(t) = f(t)$ since $W_\lambda(t) \geq 0$ in the equation (2), (10), (11), and the condition (14) is equivalent that for this equation

$$\begin{gathered} \exists \lambda_0 \in C, \quad \alpha, \beta \in \bar{\mathcal{I}}, \quad 0 \in [\alpha,\beta], \quad \text{the number} \quad \delta > 0: \\ p^{-1}(t,\ \lambda_0) \in B(\mathcal{H}), \quad \forall t \in [\alpha,\beta], \quad \left(\Delta_{\lambda_0}(\alpha,\ \beta)\, g,\, g\right) \geq \delta \left\|g\right\|^2, \quad g \in \mathcal{H}^p. \end{gathered} \tag{22}$$

Therefore, in view of [20], fulfilment of (14) implies its fulfilment with $\delta(\lambda) > 0$ instead of the δ for $\lambda \in C$ such that $p^{-1}(t, \lambda) \in B(\mathcal{H})$, $\forall\, t \in [\alpha,\ \beta]$.

Example 1. Let $l[y]$ be a symmetric 2×2 -matrix differential operation of the second order with a leading coefficient $\operatorname{diag}(p(t),\ 0)$, where $p(t) \neq 0$, and

$$m[y] = -\left(\begin{pmatrix} 0 & 0 \\ 0 & q(t) \end{pmatrix} y'\right)' + P(t)y,$$

where $q(t) > 0$, the operator $P(t) = P^*(t) > 0$. In this case, $\det p\,(t,\ \lambda) \neq 0\,(\Im\lambda \neq 0)$, $W_\lambda(t) \geq 0$ $(\Im\lambda \neq 0)$ and (14) holds, although $\det p_1(t) \equiv \det \tilde{p}_1(t) \equiv \det W_\lambda(t) \equiv 0$.

Definition 2. Every characteristic operator of the equation (2), (10), (11) corresponding to the equation (1) is said to be a characteristic operator of the equation (1) on $\mathcal{I}$ (or simply c.o.).

Lemma 3. 1^0. *We establish a correspondence between the vector-function* $f(t) \in C^s\left(\bar{\mathcal{I}}\right)$ *and the vector-function* $F_{\bar\lambda}(t)$ *that is obtained from* (16) *with* $\bar\lambda$ *instead of* λ.

Then equation (1) *is equivalent to equation* (2) *with coefficients* (10), (11), *weight* (13) *and with* $F(t) = F_{\bar\lambda}(t)$ *for such* λ *and* t *that* $p^{-1}(t,\ \lambda) \in B(\mathcal{H})$. *Namely, if* $y(t)$ *is* *a solution of the equation* (1), *then*

$$
x(t) = x\,(t,\ \lambda,\ f)
$$

$$
(23)\qquad = \begin{cases} \left(\sum_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \left(\sum_{j=1}^{n-1} \oplus \left(y^{[r-j]}\,(t\,|l-\lambda m\,) - \right.\right. \\ \qquad \left.\left. -f^{[s-j]}\,(t\,|m\,)\right)\right) \oplus y^{[n]}\,(t\,|l-\lambda m\,)\,, \quad r = 2n > s, \\ \left(\sum_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \left(\sum_{j=1}^{n} \oplus \left(y^{[r-j]}\,(t|l-\lambda m) - \right.\right. \\ \qquad \left.\left. -f^{[s-j]}\,(t|m)\right)\right) \oplus \left(-iy^{(n)}(t)\right), \quad r = 2n+1 > s, \quad r > 1, \\ y(t), \quad r = 1, \\ \left(\sum_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \left(\sum_{j=1}^{n} \oplus \left(y^{[s-j]}\,(t|l-\lambda m)\ - \right.\right. \\ \qquad \left.\left. -\ f^{[s-j]}\,(t|m)\right)\right), \quad r \le s = 2n \end{cases}
$$

$$
(\textit{here}\ \ f^{[k]}\,(t|m) \equiv 0 \ \ \textit{as}\ \ k \le 0)
$$

is a solution of (2) *with coefficients* (10), (11), *weight* (13) *and with* $F(t) = F_{\bar\lambda}(t)$. *Any solution of the equation* (2) *with coefficients* (10), (11), *weight* (13) *and with such* $F(t)$ *is equal to* (23), *where* $y(t)$ *is a solution of* (1).

2^0. *Let* $M\,(\lambda)$ *be a c.o. of the equation* (1), $\mathcal{H}^p$*-valued vector-function* $F(t) \in L^2_{W_\lambda}(\mathcal{I})$ (*in particular, one can set* $F(t) = F_{\bar\lambda}(t)$, *where* $f(t) \in C^q\left(\bar{\mathcal{I}}\right)$, $m\,[f(t),\ f(t)] < \infty$). *Then the integral* (3) *converges strongly and*

$$
(24)\qquad \|\mathcal{R}_\lambda F(t)\|^2_{L^2_{W_\lambda}(\mathcal{I})} \le \Im\,(\mathcal{R}_\lambda F,\ F)_{L^2_{W_\lambda}(\mathcal{I})}\,/\Im\lambda \quad (\Im\lambda \ne 0).
$$

If, additionally, $F(t)$ *has compact support, then the inequality* (24) *is valid without the requirement* (14).

Proof. 1^0 is verified using direct calculations taking into account (7)–(11) and Lemma 1.

2^0 is proved in [20] for the general equation (2) satisfying a condition of type (22) as $\lambda_0 \in \mathcal{A}$, and therefore it is proved for the equation (2) with coefficients (10), (11), weight (13). The statement 2^0 for $F(t)$ with compact support is also proved in [20] for the general equation (2) without a condition of the type (22). Lemma is proved. □

We notice that one can see from the proof that item 1° of Lemma 3 is valid without the fulfilment of the condition $W_\lambda(t) \ge 0\,(\Im\lambda \ne 0)$ and (14).

One can deduce from Lemmas 1–3 the following.

Corollary 1. *Let the vector-functions* $x(t),\ y(t) \in C^p\,([\alpha,\ \beta])\,,\ f(t), g(t) \in C^s\,([\alpha,\ \beta])$, $p^{-1}(t,\lambda),\ p^{-1}(t,\mu) \in B(\mathcal{H})\ \forall t \in [\alpha,\beta] \subseteq \bar{\mathcal{I}}$ *and*

$$
l[y] - \lambda m[y] = m\,[f]\,, \quad l\,[x] - \mu m\,[x] = m\,[g]\,.
$$

Then Green's formula is valid,

$$
m\,[\chi_{\alpha,\,\beta} f(t),\ \chi_{\alpha,\,\beta} x(t)] - m\,[\chi_{\alpha,\,\beta} y(t),\ \chi_{\alpha,\,\beta} g(t)] + (\lambda - \bar\mu) m\,[\chi_{\alpha,\,\beta} y(t),\ \chi_{\alpha,\,\beta} x(t)]
$$
$$
= \left([i\Re Q(t)]\,x\,(t,\ \lambda,\ f)\,,\,y\,(t,\ \mu,\ g)\right)\Big|_\alpha^\beta\,,
$$

where $x\,(t,\ \lambda,\ f)$ *is defined by* (23), $y\,(t,\ \mu,\ g)$ *is defined in a similar way using* $x(t)$ *instead of* $y(t)$, g *instead of* f *and quasi-derivatives that correspond to the expression* $l\,[x] - \mu m\,[x]$, $Q(t)$ *is defined by* (10), (11).

We consider pre-Hilbert spaces $\overset{\circ}{H}$ and H of vector-functions $y(t) \in C^s_0\left(\bar{\mathcal{I}}\right)$ and $y(t) \in C^s\left(\bar{\mathcal{I}}\right)$, $m\,[y(t),\ y(t)] < \infty$, correspondingly, with the scalar product

$$
(25)\qquad (f(t),\ g(t))_m = m\,[f(t),\ g(t)]\,,
$$

where $m\,[f(t),\ g(t)]$ is defined by (15).

Definition 3. By $\overset{\circ}{L}{}^2_m(\mathcal{I})$ and $L^2_m(\mathcal{I})$ we denote the completions of the spaces $\overset{\circ}{H}$ and H in the norms $\|\cdot\|_m = \sqrt{(\cdot\,,\,\cdot)_m}$ correspondingly. By $\overset{\circ}{\mathrm{P}}$ we denote the orthogonal projection in $L^2_m(\mathcal{I})$ onto $\overset{\circ}{L}{}^2_m(\mathcal{I})$.

We consider, in $L^2_m(\mathcal{I})$, the symmetric relation

$$\mathcal{L}'_0 = \Big\{\{\tilde{y}(t),\ \tilde{g}(t)\}|\tilde{y}(t) \overset{L^2_m(\mathcal{I})}{=} y(t),\ \tilde{g}(t) \overset{L^2_m(\mathcal{I})}{=} g(t), \quad y(t) \in C^p_0\left(\bar{\mathcal{I}}\right),\ g(t) \in C^s_0\left(\bar{\mathcal{I}}\right),\ l[y] = m\,[g]\Big\}. \tag{26}$$

Further we assume that $\mathcal{L}'_0$ consists of pairs of the type $\{y(t),\ g(t)\}$. We denote $\mathcal{L}_0 = \overline{\mathcal{L}'}_0$.

In the following theorem the generalized resolvents $R_\lambda = \int_{-\infty}^{\infty} \frac{dE_\mu}{\mu-\lambda}$ of the relation $\mathcal{L}_0$ are constructed and corresponding generalized spectral families $E_\mu = E_{\mu-0}$ [10] are found. In this theorem we denote $E_{\alpha,\beta} = \frac{1}{2}\left(E_{\beta+0} + E_\beta - E_{\alpha+0} - E_\alpha\right),\ -\infty < \alpha \le \beta < \infty$.

Theorem 1. 1^0. *Let $M(\lambda)$ be the characteristic operator of the equation* (1),

$$x_\lambda\left(t, F_{\bar{\lambda}}\right) = \mathrm{col}\left\{y_j\left(t, \lambda, f\right)\right\}_{j=1}^p \quad (y_j \in \mathcal{H}), \tag{27}$$

be the corresponding solution (3) *of the equation* (2) *with coefficients* (10), (11), *weight* (13) *and $F(t) = F_{\bar{\lambda}}(t)$, where $F_{\bar{\lambda}}(t)$ is defined by* (16) *with $\bar{\lambda}$ instead of λ, $f(t) \in C^s\left(\bar{\mathcal{I}}\right)$, $m\,[f(t),\ f(t)] < \infty$ (and therefore $F_{\bar{\lambda}}(t) \in L^2_{W_\lambda}(\mathcal{I})$ in view of* (20)*).*

Then an integro-differential operator $R_\lambda f = y_1\left(t, \lambda, f\right)$ which is densely defined in $L^2_m(\mathcal{I})$ and given by the first vector-valued component of the solution (27) *is, after closing, the generalized resolvent of the relation $\mathcal{L}_0$.*

2^0. *Let $M(\lambda)$ be the characteristic operator of the equation* (1) *(and therefore by* [20] *$\Im M(\lambda) \ge 0$ as $\Im\lambda > 0$) and $\sigma(\mu) = w-\lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_0^\mu \Im \mathrm{M}(\mu + i\varepsilon)\,d\mu$ be the spectral operator-function that corresponds to $M(\lambda)$.*

Let E_μ be the generalized spectral family corresponding to the generalized resolvent R_λ from the item 1° of this theorem. Then for any $f(t) \in C^s_0\left(\bar{\mathcal{I}}\right)$ the equality

$$\overset{\circ}{P} E_{\alpha,\beta} f(t) = \overset{\circ}{P} \int_\alpha^\beta \left[X_\mu(t)\right]_1 d\sigma(\mu)\, \varphi(\mu, f), \tag{28}$$

is valid in $\overset{\circ}{L}{}^2_m(\mathcal{I})$, where $\left[X_\lambda(t)\right]_1 \in B\left(\mathcal{H}^p, \mathcal{H}\right)$ is the first row of the operator solution $X_\lambda(t)$ of the homogeneous equation (2) *with coefficients* (10), (11) *that is written in the matrix form and such that $X_\lambda(0) = I_p$,*

$$\varphi(\mu, f) = \int_{\mathcal{I}} \left(\left[X_\mu(t)\right]_1\right)^* m\,[f]\,dt, \tag{29}$$

if $p^{-1}(t, \mu) \in B(\mathcal{H})\ \forall t \in \bar{\mathcal{I}},\ \mu \in [\alpha, \beta]$.

Moreover, for $f(t) \in \mathrm{D}\left(\mathcal{L}'_0\right)$ (see (26)) and with $r > s$ (or with $r < s$, if additionally $\overset{\circ}{P}\left(E_{+0} - E_0\right) f(t) = 0$), the inverse formula in $\overset{\circ}{L}{}^2_m(\mathcal{I})$

$$f(t) = \overset{\circ}{P} \int_{-\infty}^\infty \left[X_\mu(t)\right]_1 d\sigma(\mu)\, \varphi(\mu, f), \tag{30}$$

and Parceval's equality

$$m\,[f(t), g(t)] = \left(\varphi(\mu, f),\ \varphi(\mu, g)\right)_{L^2(R,\,d\sigma)}, \tag{31}$$

are valid, where $g(t) \in C^s_0\left(\bar{\mathcal{I}}\right)$.

Let us explain that, for $r > s$,

$$\overset{\circ}{P} \int_{-\infty}^\infty = \lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}} \overset{\circ}{P} \int_\alpha^\beta$$

in (30), and for $r < s$,

$$\overset{\circ}{P}\int_{-\infty}^{\infty} = \lim_{\substack{\alpha\to-\infty\\ \beta\to-0}} \overset{\circ}{P}\int_{\alpha}^{\beta} + \lim_{\substack{\delta\to\infty\\ \gamma\to+0}} \overset{\circ}{P}\int_{\gamma}^{\delta},$$

where the limits exist in $\overset{\circ}{L^2_m}(\mathcal{I})$. Similarly, $\int_{-\infty}^{\infty} = \int_{-\infty}^{-0} + \int_{+0}^{\infty}$ in the right-hand side of (31) for $r < s$.

Proof. Let for definiteness $r \le s = 2n$ (for $r > s$ the proof becomes simpler due to (10)–(12)).

1^0. Let $\Im\lambda \neq 0$. In view of the item 1° of Lemma 3, $y_1(t, \lambda, f)$ is a solution of (1). Using (10) and Lemmas 1–3 one can show that

$$\begin{aligned}\sum_{k=0}^{s} m_k\left[y_1(t, \lambda, f), y_1(t, \lambda, f)\right] - \Im\left(\sum_{k=0}^{s} m_k\left[y_1(t, \lambda, f), f(t)\right]\right)/\Im\lambda \\ = \big(W_\lambda(t)x(t,\lambda,f), x(t,\lambda,f)\big)_{\mathcal{H}^s} - \Im\big(W_\lambda(t)x(t,\lambda,f), F_{\bar\lambda}(t)\big)_{\mathcal{H}^s}/\Im\lambda,\end{aligned} \tag{32}$$

although for $r \le s$ the corresponding items in the right- and left-hand sides of (32) do not coincide. Therefore[1]

$$\begin{aligned}\|y_1(t, \lambda, f)\|^2_{L^2_m(\alpha,\beta)} - \Im\left(y_1(t, \lambda, f), f(t)\right)_{L^2_m(\alpha,\beta)}/\Im\lambda \\ = \|x(t, \lambda, f)\|^2_{L^2_{W_\lambda}(\alpha,\beta)} - \Im\left(x(t, \lambda, f), F_{\bar\lambda}(t)\right)_{L^2_{W_\lambda}(\alpha,\beta)}/\Im\lambda.\end{aligned} \tag{33}$$

In view of the item 2° of Lemma 3 a nonnegative limit of the right-hand-side of (33) exists, when $(\alpha, \beta) \uparrow \mathcal{I}$. Consequently

$$\|y_1(t, \lambda)\|^2_{L^2_m(\mathcal{I})} \le \Im\left(y_1(t, \lambda), f(t)\right)_{L^2_m(\mathcal{I})}/\Im\lambda. \tag{34}$$

Since $M(\lambda) = M^*(\bar\lambda)$, then the operator $\mathcal{R}_\lambda$ (3) in $L^2_{W_\lambda}(\alpha, \beta)$ with finite $(\alpha, \beta) \subseteq \mathcal{I}$ possesses the property $\mathcal{R}_\lambda = \mathcal{R}^*_{\bar\lambda}$. Therefore $([\Re Q(t)]\, R_\lambda F, R_{\bar\lambda} G)|_\alpha^\beta = 0$. It follows from Corollary 1 and (34) that $\forall f(t), g(t) \in C^s(\bar{\mathcal{I}}) \bigcap L^2_m(\mathcal{I})$

$$m\left[y_1(\lambda, f), g\right] = m\left[f, y_1(\bar\lambda, g)\right].$$

Thus the closure of the operator $R_\lambda f = y_1(t, \lambda, f)$ in $L^2_m(\mathcal{I})$ possesses a property

$$R_\lambda = R^*_{\bar\lambda}. \tag{35}$$

Since in view of (34) for any $f(t), g(t) \in C^s(\bar{\mathcal{I}}) \bigcap L^2_m(\mathcal{I})$ and with $(\alpha, \beta) \uparrow \mathcal{I}$,

$$\left(y_1(\lambda, f), g\right)_{L^2_m(\alpha,\beta)} \to \left(y_1(\lambda, f), g\right)_{L^2_m(\mathcal{I})}$$

uniformly in λ from any compact set $\in C/R$, we see that, in view of analyticity of the operator function $M(\lambda)$ and vector-function $W_\lambda(t)F_{\bar\lambda}(t)$, (17), the operator R_λ depends analytically on the non-real λ in view of [15, p. 195].

Finally, similarly to the case $s = 0$ [20] using Corollary 1 it is verified that

$$R_\lambda(\mathcal{L}_0 - \lambda) \subset \mathbf{I}, \tag{36}$$

where $\mathbf{I}$ is the graph of the identical operator in $L^2_m(\mathcal{I})$.

Taking into account (34)–(36) and analyticity of R_λ, we see in view of [10] that R_λ is a generalized resolvent of $\mathcal{L}_0$. Item 1° is proved.

---

[1] In particular this implies that

$$\|y_1(t,\lambda,f)\|^2_{L^2_m(\alpha,\beta)} - \frac{\Im\left(y_1(t,\lambda,f), f(t)\right)_{L^2_m(\alpha,\beta)}}{\Im\lambda} = \frac{U\left[x(\beta,\lambda,f)\right] - U\left[x(\alpha,\lambda,f)\right]}{2\Im\lambda}.$$

2^0. Let the vector-functions $f(t),\ g(t)\in C_0^s(\mathcal{I}),\ \lambda=\mu+i\varepsilon,\ G_\lambda(t)$ be defined by (16) with $g(t)$ instead of $f(t)$. In view of the Stieltjes inversion formula,

$$\begin{aligned}(E_{\alpha,\beta}f,\ g)_m &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\alpha^\beta \left(\left[y_1(\lambda,\ f)-y_1\left(\bar\lambda,\ f\right)\right],\ g\right)_m d\mu \\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\alpha^\beta \Big[\left(x\left(t,\lambda,\ f\right),\ G_\lambda(t)\right)_{L^2_{W_\lambda}(\mathcal{I})}-\left(x\left(t,\bar\lambda,\ f\right),\ G_{\bar\lambda}(t)\right)_{L^2_{W_\lambda}(\mathcal{I})} \\ &+2i\int_{\mathcal{I}}\Big(\left(\Im p^{-1}(t,\lambda)\right)f^{[n]}(t|m),\ g^{[n]}(t|m)\Big)dt\Big]d\mu \\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\alpha^\beta\Big[\Big(M(\lambda)\int_{\mathcal{I}}X^*_{\bar\lambda}(t)W_\lambda(t)F_{\bar\lambda}(t)dt,\ \int_{\mathcal{I}}X^*_\lambda(t)W_{\bar\lambda}(t)G_\lambda(t)dt\Big) \\ &-\Big(M^*(\lambda)\int_{\mathcal{I}}X^*_\lambda(t)W_{\bar\lambda}(t)F_\lambda(t)dt,\ \int_{\mathcal{I}}X^*_{\bar\lambda}(t)W_\lambda(t)G_{\bar\lambda}(t)dt\Big)\Big]d\mu \\ &= \int_\alpha^\beta\Big(d\sigma(\mu)\int_{\mathcal{I}}X^*_\mu(t)W_\mu(t)F_\mu(t)dt,\ \int_{\mathcal{I}}X^*_\mu(t)W_\mu(t)G_\mu(t)dt\Big),\end{aligned}\tag{37}$$

where the second equality is a corollary of (10), (13), (20), (23), next to last is a corollary of (3), and the last follows from the well-known generalization of the Stieltjes inversion formula [27, proposition (Б), p. 803], [4, Lemma, p. 952]. But for $\mu\in[\alpha,\beta]$

$$\int_{\mathcal{I}}X^*_\mu(t)W_\mu(t)F_\mu(t)dt=\int_{\mathcal{I}}\left([X_\mu(t)]_1\right)^* m[f]\,dt,\tag{38}$$

because, in view of (20),

$$\begin{aligned}\forall h\in\mathcal{H}^{\mathrm{s}}:\quad &\Big(\int_{\mathcal{I}}X^*_\mu(t)W_\mu(t)F_\mu(t)dt,\ h\Big)\\ &=\int_{\mathcal{I}}(W_\mu(t)F_\mu(t),\ X_\mu(t)h)=\int_{\mathcal{I}}\Big(\left([X_\mu]_1\right)^* m[f],\ h\Big)dt.\end{aligned}$$

Due to (37), (38),

$$(E_{\alpha,\beta}f,\ g)_m=\int_\alpha^\beta\left(d\sigma(\mu)\varphi(\mu,\ f),\ \varphi(\mu,\ g)\right).\tag{39}$$

Replacing $\int_\alpha^\beta$ in (39) by an integral sum and using (20), (38) we obtain that

$$\begin{aligned}(E_{\alpha,\beta}f,\ g)_m&=\Big(\int_\alpha^\beta[X_\mu(t)]_1\,d\sigma(\mu)\varphi(\mu,\ f),\ g(t)\Big)_m\\ &=\Big(\mathring{P}\int_\alpha^\beta[X_\mu(t)]_1\,d\sigma(\mu)\varphi(\mu,\ f),\ g(t)\Big)_m\end{aligned}$$

and (28) is proved.

Since $E_\infty f(t)=f(t)$ if $f(t)\in D(\mathcal{L}_0')$, passing to the limit in (28), (39) for $\alpha\to-\infty,\ \beta\to-0$ and $\alpha\to+0,\ \beta\to\infty$ we obtain (30) and (31). Item 2° and Theorem 1 are proved. □

The following remark follows from [5, 6] and from [20, formula (1.70)].

Remark 1. If $m[y]=w(t)y$ and if for the equation (2) with coefficients (10), (11) that corresponds to the equation (6), the condition

$$\exists\lambda_0\in C,\quad \alpha,\ \beta,\ \delta>0:\ \left(\Delta_{\lambda_0}(\alpha,\ \beta)g,\ g\right)\ge\delta\|g\|^2\quad\forall g\in N^\perp\tag{40}$$

holds true, then $R_\lambda f$ for any generalized resolvent R_λ of $\mathcal{L}_0$ and any $f(t)\in C_0(\bar{\mathcal{I}})$ have the same representation as in item 1° of Theorem 1.

An analysis of the proof of Theorem 1 shows the following.

Remark 2. If (14) is not assumed to hold, then we have the following: 1) item 1° of Theorem 1 is valid either for $f(t) \in C^s\left(\bar{\mathcal{I}}\right)$, if the interval $\mathcal{I}$ is finite or for $f(t) \in C_0^s(\mathcal{I})$, if $\overset{\circ}{L}{}_m^2(\mathcal{I}) = L_m^2(\mathcal{I})$.

2) Identity (28) holds for $L_m^2(\mathcal{I})$, if one changes it as follows: a) $\sigma(\mu)$ is a spectral function corresponding to $PM(\lambda)P$ ($\Im PM(\lambda)P \geq 0$ as $\Im\lambda > 0$ [20]); b) remove $\overset{\circ}{P}$ from (28); c) $f(t) \in C^s\left(\bar{\mathcal{I}}\right)$ and $\varphi(\mu, f) = \int_{\mathcal{I}} X_\mu^*(t) W_\mu(t) F_\mu(t)dt$, if the interval $\mathcal{I}$ is finite or $f(t) \in C_0^s\left(\bar{\mathcal{I}}\right)$, if $\overset{\circ}{L}{}_m^2(\mathcal{I}) = L_m^2(\mathcal{I})$.

The following theorem establishes a relationship between the generalized resolvents of the relations $\mathcal{L}_0$ that are given by Theorem 1, and the boundary value problems for the equation (1) with boundary conditions depending on the spectral parameter. Already in the simplest case, where l and m that generate (1) are self-adjoint differential operators we see that the pair $\{y,\ f\}$ satisfies the boundary conditions that contain both y derivatives and f derivatives of corresponding orders at the ends of interval.

Theorem 2. *Let the interval $\mathcal{I} = (a,\ b)$ be finite.*

Let the operator-functions $\mathcal{M}_\lambda,\ \mathcal{N}_\lambda \in B(\mathcal{H}^p)$, depend analytically on the non-real λ,

$$\mathcal{M}_{\bar\lambda}^*\left[\Re Q(a)\right]\mathcal{M}_\lambda = \mathcal{N}_{\bar\lambda}^*\left[\Re Q(b)\right]\mathcal{N}_\lambda \quad (\Im\lambda \neq 0), \tag{41}$$

where $Q(t)$ is the coefficient of the equation (2) *corresponding by Lemma 3 to the equation* (1) (*see* (*10*), (11)),

$$\|\mathcal{M}_\lambda h\| + \|\mathcal{N}_\lambda h\| > 0 \quad (0 \neq h \in \mathcal{H}^p,\ \Im\lambda \neq 0), \tag{42}$$

the lineal $\{\mathcal{M}_\lambda h \oplus \mathcal{N}_\lambda h \,|h \in \mathcal{H}^p\} \subset \mathcal{H}^{2p}$ is a maximal $\mathcal{Q}$-nonnegative subspace since $\Im\lambda \neq 0$, where $\mathcal{Q} = (\Im\lambda)\operatorname{diag}(\Re Q(a), -\Re Q(b))$ (*and therefore*

$$\Im\lambda\left(\mathcal{N}_\lambda^*\left[\Re Q(b)\right]\mathcal{N}_\lambda - \mathcal{M}_\lambda^*\left[\Re Q(a)\right]\mathcal{M}_\lambda\right) \leq 0 \quad (\Im\lambda \neq 0)). \tag{43}$$

Then for any $f(t) \in C^s\left(\bar{\mathcal{I}}\right)$ the boundary problem that is obtained by adding the boundary conditions

$$\exists h = h(\lambda,\ f) \in \mathcal{H}^p:\ x(a,\lambda, f) = \mathcal{M}_\lambda h, \quad x(b,\ \lambda,\ f) = \mathcal{N}_\lambda h, \tag{44}$$

to the equation (1)*, where $x(t,\ \lambda,\ f)$ is defined by* (23)*, has the unique solution $R_\lambda f$ as $\Im\lambda \neq 0$. It is generated by the generalized resolvent R_λ of the relation $\mathcal{L}_0$ that is constructed, as in item* 1° *of Theorem 1, using the c.o.*

$$M(\lambda) = -\frac{1}{2}\left(X_\lambda^{-1}(a)\mathcal{M}_\lambda + X_\lambda^{-1}(b)\mathcal{N}_\lambda\right)\left(X_\lambda^{-1}(a)\mathcal{M}_\lambda - X_\lambda^{-1}(b)\mathcal{N}_\lambda\right)^{-1}(iG)^{-1},$$

where

$$\left(X_\lambda^{-1}(a)\mathcal{M}_\lambda - X_\lambda^{-1}(b)\mathcal{N}_\lambda\right)^{-1} \in B(\mathcal{H}^p) \quad (\Im\lambda \neq 0),$$

$X_\lambda(t)$ is an operator solution of the homogeneous equation (2) *with coefficients* (10)*,* (11) *and such that $X_\lambda(0) = I_p$.*

Proof. Proof follows from Lemma 3, Theorem 1 and from from [20, Remark 1.1]. □

For $s = 0$, Theorem 2 is known (see [28, 5] as $\dim\mathcal{H} < \infty$, [20] as $r = 1$, $\dim\mathcal{H} = \infty$).

Example 2. Let, in the equation (1), $r = 4,\ s = 2$.

a) Let $\mathcal{M}_\lambda = \mathcal{N}_\lambda = I_p$. Then the boundary conditions (44) can be represented in the form

$$\begin{gathered} y(a) = y(b), \quad y'(a) = y'(b), \\ y^{[2]}(a\,|l) = y^{[2]}(b\,|l), \\ y^{[3]}(a\,|l - \lambda m) - f^{[1]}(a\,|m) = y^{[3]}(b\,|l - \lambda m) - f^{[1]}(b\,|m). \end{gathered} \tag{45}$$

In particular for the equation

$$y^{(IV)} - \lambda(-y'' + y) = -f'' + f \tag{46}$$

conditions (45) have the form

$$y(a) = y(b), \quad y'(a) = y'(b), \quad y''(a) = y''(b), \\ y'''(a) + f'(a) = y'''(b) + f'(b). \tag{47}$$

b) If $\dim \mathcal{H} = 1$,

$$\mathcal{M}_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{N}_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then the boundary conditions (44) can be written in the form

$$y(a) = y(b) = 0, \quad y'(a) = y^{[2]}(b|l), \quad y^{[2]}(a|l) = -y'(b). \tag{48}$$

In particular for the equation (46) conditions (48) have the form

$$y(a) = y(b) = 0, \quad y'(a) = y''(b), \quad y''(a) = -y'(b). \tag{49}$$

On functions satisfying either the boundary condition (47) with $f(t) \equiv 0$ or the boundary conditions (49), expressions $l[y] = y^{(\mathrm{IV})}$ and $m[y] = -y'' + y$ define a self-adjoint and symmetric operators correspondingly.

When the boundary conditions are such that $l[y]$ defines a self-adjoint operator and $m[y]$ defines only a symmetric operator on functions satisfying these conditions, then the eigenfunction expansion of the special scalar equation (6) in the regular case is constructed in [7, 14]. We note that, in the general case, the boundary conditions (44) are not reduced to the boundary condition of [7, 14] type, since it is possible that conditions (44) in the case [7, 14] do not imply s boundary conditions containing only the derivatives of order up to $s-1$.

In the next theorem, $\mathcal{I} = R$ and condition (14) hold both on the negative semi-axis R_- (i.e. as $\mathcal{I} = R_+$) and on the positive semi-axis R_+ (i.e. as $\mathcal{I} = R_-$).

Theorem 3. *Let $\mathcal{I} = R$, the coefficient of the equation* (6) *be periodic on each of the semi-axes R_+ and R_- with periods $T_+ > 0$ and $T_- > 0$ correspondingly. Then the spectrums of the monodromy operators $X_\lambda(\pm T_\pm)$ ($X_\lambda(t)$ is from Theorem 2) do not intersect the unit circle as $\Im\lambda \neq 0$, the c.o. $M(\lambda)$ of the equation* (1) *is unique and equal to*

$$M(\lambda) = \left(\mathcal{P}(\lambda) - \frac{1}{2} I_p\right) (iG)^{-1} \quad (\Im\lambda \neq 0) \tag{50}$$

where the projection $\mathcal{P}(\lambda) = P_+(\lambda)(P_+(\lambda) + P_-(\lambda))^{-1}$, $P_\pm(\lambda)$ are Riesz projections of the monodromy operators $X_\lambda(\pm T_\pm)$ that correspond to their spectrums lying inside the unit circle, $(P_+(\lambda) + P_-(\lambda))^{-1} \in B(\mathcal{H}^p)$ as $\Im\lambda \neq 0$.

Also let $\dim \mathcal{H} < \infty$, $\mathbf{A} = \{\mu \in R : \det p(t, \mu) \neq 0 \ \forall t \in (-T_-, T_+)\}$, a finite interval $\Delta \subset \mathbf{A}$. Then in item 2° of Theorem 1 $d\sigma(\mu) = d\sigma_{ac}(\mu) + d\sigma_d(\mu)$, $\mu \in \Delta$. Here $\sigma_{ac}(\mu) \in AC(\Delta)$ and, for $\mu \in \Delta$,

$$\sigma'_{ac}(\mu) = \frac{1}{2\pi} G^{-1} \left(Q^*_-(\mu) G Q_-(\mu) - Q^*_+(\mu) G Q_+(\mu)\right) G^{-1} \tag{51}$$

where the projections $Q_\pm(\mu) = q_\pm(\mu)(P_+(\mu) + P_-(\mu))^{-1}$, $q_\pm(\mu)$ are Riesz projections of the monodromy matrixes $X_\mu(\pm T_\pm)$ corresponding to the multiplicators equal to 1 *such that they are shifted inside the unit circle as μ is shifted to the upper half plane, $P_\pm(\mu) = P_\pm(\mu + i0)$); $\sigma_d(\mu)$ is a jump function.*

Let us notice that the sets on which $q_\pm(\mu)$, $P_\pm(\mu)$, $(P_+(\mu) + P_-(\mu))^{-1}$ are not infinitely differentiable do not have finite limit points $\in \mathbf{A}$ as well as the set of points of increase of $\sigma_d(\mu)$.

Proof. Let the operator G be indefinite (otherwise the proof is modified in an obvious way). The unitary dichotomy of the operators $X_\lambda(\pm T_\pm)$ and the fact that $M(\lambda)$ (50) is a c.o. of the equation (1) on $(-T_-,\ T_+)$ follow from [20, p. 161, 162]. Since $X_\lambda(t \pm T_\pm) = X_\lambda(t)X_\lambda(\pm T_\pm)$, $t \in R_\pm$, and $\Im\lambda U[X_\lambda(t)]$ does not decrease as $\Im\lambda \neq 0$, we have that $M(\lambda)$ (50) is a c.o. of the equation (1) on any finite $\mathcal{I}$ and therefore it is a c.o. on the axis.

Let for some non-real λ_0 the homogeneous equation (2) with coefficients (10), (11) have a solution $x(t) \in L^2_{W_{\lambda_0}}(R^1)$.

Since, for $k \in \mathrm{Z}_+$,

$$\|x(t)\|^2_{L^2_{W_{\lambda_0}}(R^1)} = \sum_{j=-\infty}^{0} \left(\Delta_{\lambda_0}(-kT_-,\ 0)\, x(-jkT_-),\, x(-jkT_-)\right) + \sum_{j=0}^{\infty} \left(\Delta_{\lambda_0}(0,\ kT_+)\, x(jkT_+),\ x(jkT_+)\right),$$

and using condition (14) on $\mathcal{I} = R_\pm$ and estimates of the type [8, p. 290], we see that $x(0) \in H_- \bigcap H_+$, where $H_\pm$ are the invariant subspaces of the operators $X_\lambda(\pm T_\pm)$ that correspond to their spectrums lying inside the unit circle. But $H_- \bigcap H_+ = \{0\}$ [20, p. 162]. Therefore in view of Lemma 1.5 from [20] the c.o. $M(\lambda)$ (50) is unique.

Formula (51) follows from [19, Theorem 13]. Decomposition $d\sigma(\mu) = d\sigma_{ac}(\mu) + d\sigma_d$, $\mu \in \Delta$ as well as the remark after the formulating of Theorem 3 are proved in the same way as similar statements in [18]. In the proof for $r \leq s$ one should take into account that Krein-Lyubarsky theory [22] for homogeneous periodic system (2) is still valid for $\lambda \in \mathcal{A} \bigcap R$ and when $H_\lambda(t)$ contains λ is a Nevanlinna manner; it can be seen analysing the statement of this theory in [29, p. 147–150, 181–183] and the proof of Theorem 1.2 from [12, p. 305]. Theorem is proved. □

Example 3. Let $\dim \mathcal{H} = 1$, $l[y] = (i)^n y^{(n)}$, $m[y] = (i)^{2n} y^{(2n)} + y$, $\mathcal{I} = R$ (and therefore $L^2_m(\mathcal{I}) = \overset{\circ}{L}{}^2_m(\mathcal{I})$). In this case, $E_0 = E_{+0}$, the spectral matrix $\sigma(\mu) \in AC_{loc}$, and, in view of Theorem 3 for $n = 1, 3, \ldots$,

$$\sigma'(\mu) = \frac{1}{2\pi n\,(k^{2n}+1)}\Big\{2k^n A^n + (1-k^{2n})I_{2n} + \sum_{j=1}^{n-1}\big(k^{2n-j} + (-1)^{j+1}k^j\big)\big(A^j + (-1)^{j+1}A^{-j}\big)\Big\}(iJ)^{-1}, \tag{52}$$

since $|\mu| < \frac{1}{2}$, where $\sum_{j=1}^{0} = 0$, $rg\sigma'(\mu) = 2$, and $\sigma'(\mu) = 0$ as $|\mu| > \frac{1}{2}$. For $n = 2,\ 4,\ 6,\ \ldots$, one has

$$\sigma'(\mu) = \frac{1}{\pi n\,(k^{2n}-1)}\sum_{j=1}^{n/2}\big(k^{2n-2j+1} - k^{2j-1}\big)\big(A^{2j-1} + A^{1-2j}\big)(iJ)^{-1} \tag{53}$$

since $0 < \mu < \frac{1}{2}$, where $rg\sigma'(\mu) = 4$, and $\sigma'(\mu) = 0$ as $\mu \notin [0, 1/2]$. In (52), (53) $k = -i\sqrt[n]{\frac{1+(-1)^n\sqrt{1-4\mu^2}}{2\mu}}$, $A = (iJ)^{-1} H_\mu(t)$, where the matrices J and $H_\mu(t)$ are independent of t and defined by (10).

In particular, for $n = 1$, $|\mu| < 1/2$,

$$\sigma'(\mu) = \frac{1}{2\pi}\begin{pmatrix} \frac{2}{\sqrt{1-4\mu^2}} & 0 \\ 0 & \frac{1}{2}\sqrt{1-4\mu^2} \end{pmatrix}.$$

And for $n = 2,\ 0 < \mu < \frac{1}{2}$,

$$\sigma'(\mu) = \frac{1}{\pi}\sqrt{\frac{1+\sqrt{1-4\mu^2}}{2\mu(1-2\mu)}} \cdot \frac{1}{\sqrt{1-2\mu}+\sqrt{1+2\mu}} \begin{pmatrix} 1 & 0 & 0 & \mu \\ 0 & 1 & \mu & 0 \\ 0 & \mu & \mu(1-\mu) & 0 \\ \mu & 0 & 0 & \mu(1-\mu) \end{pmatrix}.$$

Remark 3. In the case $r \leq s$ in contrast to the case $r > s$, the point spectrum of the relation $\mathcal{L}'_0$ can be non-empty including the case when $\mathcal{L}'_0$ corresponds to the scalar equation (1) with periodic coefficients on the axis.

Indeed let $m[y] = -y'' + y,\ l[y] = p(t)y$, where $p(t+4) = p(t) \in C(R),\ p(t) = \begin{cases} 1, & 4k \leq t \leq 4k+1 \\ 0, & 4k+2 \leq t \leq 4k+3 \end{cases}$. Then for any function $y(t) \in C_0^2(R)$ such that $\operatorname{supp} y(t) \subset \bigcup_k [4k+2, 4k+3]$, the pair $\{y(t),\ 0\} \in \mathcal{L}'_0$, i.e. the point spectrum of $\mathcal{L}'_0$ contains $\lambda = 0$. Similarly an example for $r = 1,\ r = 2,\ s = 2$ is constructed.

The following remark is proved similarly to Theorem 3.

Remark 4. Let $\mathcal{I} = R_+$, the coefficients of the equation (6) be periodic with the period $T > 0$. Then

1) Any c.o. of the equation (1) is found by combining the formula (4.4) from [20] and the formula (50).

2) If $\infty > \dim \mathcal{H}^p = 2k$, L is a k dimensional G-neutral subspace (see [8]), then the c.o. of the equation (1), which corresponds to a self-adjoint boundary condition in zero, $x(0,\ \lambda,\ f) \in L$, is given by the formula (36) from [18]. The corresponding spectral matrix-function $\sigma(\mu)(d\sigma_{ac}(\mu), \mu \in \Delta_+)$ is given for $r > s$ ($r \leq s$) by Theorem 6 from [18], starting from the monodromy matrix $X_\lambda(T)$ of the equation (6), where Δ_+ is an analog of Δ from Theorem 3.

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